The $1+1$ Dimensional Abelian Higgs Model Revisited: Physical Sector and Solitons[1]

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In this paper the two dimensional abelian Higgs model is revisited. We show that in the physical sector, the solutions to the Euler–Lagrange equations include solitons.

1 Introduction

The two dimensional abelian Higgs model is revisited. Using the Dirac formalism for constrained systems, it has been established [1] that this model, in its gauge invariant physical sector, corresponds to the coupling of a pseudoscalar field, namely the electric field, with a real scalar field, as a matter of fact the Higgs field which is the radial component of the original complex scalar field in field configuration space. At the classical level, the Euler–Lagrange equations of motion lead to a system of coupled non-linear equations of which the spectrum of solutions includes solitons. The linearised spectrum of fluctuations of these solitonic solutions has been identified in order to ascertain their classical stability.

This contribution is organised as follows. Section 2 discusses the Euler–Lagrange equations of motion. Upon compactification of the spatial dimension into a circle, in Section 3 static solutions to these equations are constructed in closed analytic form in terms of the Jacobi elliptic functions. In Section 4 the linearised spectrum of fluctuations for these classical solutions is identified. Some concluding remarks are provided in Section 5.

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2 The Euler–Lagrange Equations of Motion

The $1+1$ dimensional abelian Higgs model is described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}\left[\partial_\mu A_\nu - \partial_\nu A_\mu\right]\left[\partial^\mu A^\nu - \partial^\nu A^\mu\right] + \left|\left(\partial_\mu + ieA_\mu\right)\phi\right|^2 - V(|\phi|),$$

where A_μ is the gauge field with gauge coupling constant e and ϕ a complex scalar field with self-interactions described by the U(1) gauge invariant potential $V(|\phi|)$. Choosing a parametrisation of the complex field as $\phi = \rho e^{i\varphi}/\sqrt{2}$ and through the Dirac formalism for constrained systems [2] the model in its physical sector is described by [1]

$$\mathcal{L}_{\text{phys}} = \frac{1}{2}\frac{1}{\rho^2}(\partial_\mu B)^2 - \frac{1}{2}e^2 B^2 + \frac{1}{2}(\partial_\mu \rho)^2 - V(\rho) - \partial_0[\frac{1}{\rho^2}B\partial_0 B] + \partial_1\left[B(\partial_0\varphi + eA_0)\right], \tag{1}$$

where $B = -\frac{1}{e}E$ and E is the electric field. The Euler–Lagrange equations of motion for ρ and B are, respectively,

$$\partial_\mu^2 \rho + \frac{1}{\rho^3}(\partial_\mu B)^2 + \frac{\partial V}{\partial \rho} = 0, \qquad \partial_\mu\left(\frac{1}{\rho^2}\partial^\mu B\right) + e^2 B = 0. \tag{2}$$

These equations are non-linear and coupled, and not all their solutions can be analytically found. However it might be possible to find analytical solutions for static states of finite energy. In order to reduce the difficulties of solving these equations, we consider configurations where the electric field vanishes, $E(t,x) = 0$. In fact the total energy is $E_{\text{phys}} = E_k + E_p$, where E_k is the total kinetic energy and E_p the total potential energy of the fields, with,

$$E_k = \int_{-L}^{L} dx \left(\frac{1}{2\rho^2}(\frac{\partial}{\partial t}B)^2 + \frac{1}{2}(\frac{\partial}{\partial t}\rho)^2\right),$$

and

$$E_p = \int_{-L}^{L} dx \left(\frac{1}{2}\frac{1}{\rho^2}(\frac{\partial}{\partial x}B)^2 + \frac{1}{2}e^2 B^2 + \frac{1}{2}(\frac{\partial}{\partial x}\rho)^2 + V(\rho)\right).$$

From these expressions, one notices that if B is non zero but ρ vanishes for some value of x, we have a singularity in the equations of motion and furthermore the energy may become infinite, unless the electric field vanishes faster than ρ in such a manner that the quotient of these zeros remains finite. But such behaviour induces large spatial gradients in both ρ and B, implying a large value for the total energy. Hence in order to minimise the energy, one needs to consider configurations with $B = 0$. Another advantage of this restriction is that the equations are no longer coupled and we simply have to solve

$$B = 0, \qquad \partial_\mu^2 \rho + \frac{\partial V}{\partial \rho} = 0. \tag{3}$$

A further assumption is required though, since it is still difficult to construct all the time dependent solutions to the above equation (except for those obtained by Lorentz boosts from a static solution). Moreover, any time dependence for a solution increases its total energy. Consequently we restrict further to static configurations in which ρ only depends on the space variable x, leading to the single non trivial equation,

$$B(t,x) = 0, \qquad \frac{d^2\rho(x)}{dx^2} - \frac{dV(\rho(x))}{d\rho(x)} = 0, \tag{4}$$

where for the potential energy henceforth the Higgs choice will be made, $V(\rho) = (1/8)M^2\left(\rho^2 - \rho_0^2\right)^2$, with $M > 0$ a mass scale and ρ_0 the expectation value of the scalar field.

3 Static Solutions

In order to solve the above equation, let us choose to compactify space into a circle of length $2L$ with $-L \le x \le L$. For any constant values of ρ associated to the minima of the potential, where the potential also vanishes given our choice of subtraction constant, the non-linear equation in (3) is satisfied. These constant solutions $B(t,x) = 0$, $\rho(t,x) = \pm\rho_0$ correspond to vacuum configurations, of which the total energy values are minimal and vanish.

Besides these configurations, we also have non constant static solutions which behave like solitons. Given the equation for ρ,

$$\rho'' - \frac{\partial V}{\partial \rho} = 0, \tag{5}$$

where $\rho''(x)$ stands for the double derivative with respect to x, there exists a conservation law expressed as

$$\frac{1}{2}((\rho'(x))^2 = V(\rho) + C_0, \tag{6}$$

C_0 being some integration constant. In this form the solution is readily obtained by quadrature. On the circle of length $2L$ one finds,

$$\rho_s(x) = \pm\rho_0\sqrt{\frac{2k^2}{1+k^2}}\ \mathrm{sn}\left[\frac{1}{2}M\rho_0\sqrt{\frac{2}{1+k^2}}x\right].$$

it being understood that the solutions obey either periodic or anti-periodic boundary conditions given the remaining freedom in the sign of ρ existing for the choice of polar parametrisation of the complex scalar field ϕ [1]. These solutions are thus expressed in terms of the Jacobi elliptic functions [3]. According to whether a periodic or an anti-periodic solution is obtained, the remaining single integration constant is related to the elliptic modulus k through the condition,

$$\begin{aligned} LM\rho_0 &= 4(n+r)K(k)\sqrt{\frac{1+k^2}{2}}, \qquad n \in \mathbb{N}, \\ r &= \begin{cases} 0: & \text{periodic}, \\ 1/2: & \text{antiperiodic}. \end{cases} \end{aligned}$$

Their total energy is represented through the expression

$$E_{\rm phys} = \int_{-\infty}^{+\infty} dx \left\{\frac{1}{4}M^2\left(\rho_s^2 - \rho_0^2\right)^2 + C_0\right\}.$$

By substituting the above explicit expression, one finds

$$E_{\rm phys} = \frac{LM^2\rho_0^4}{2(n+r)K(k)}\int_0^{2(n+r)K(k)} dy \left\{\left[\frac{1}{4} + \frac{k^2}{(1+k^2)^2}\right] - \frac{2k^2}{(1+k^2)}\ \mathrm{sn}^2 y + \frac{2k^4}{(1+k^2)^2}\ \mathrm{sn}^4 y\right\}.$$

This quantity is finite since $0 \leq k^2 \leq 1$ et $0 \leq\ \mathrm{sn}^2 y \leq 1$. This very fact together with the spatial profile $\rho(x)$ of these configurations justifies their interpretation as solitons. In particular the periodic solution with $n = 1$ in the decompactification limit $L \to \infty$ reduces to the celebrated kink solutions of the ϕ^4 real scalar field theory in 1+1 dimensions.

4 Spectrum of Fluctuations

Given the explicit solutions of the previous Section, in the present Section we address the issue of the classical stability under linearised fluctuations in field configuration space. This requires the computation of the spectrum of fluctuation eigenvalues, to ascertain that none of these is negative, which otherwise would establish that some modes have an unbounded above exponentially growing amplitude, spelling disaster for the corresponding solution.

Let us consider arbitrary time- and space-dependent fluctuations around the identified solutions, $B(t,x) = \delta B(t,x)$, $\rho(t,x) = \rho_s(x) + \delta\rho(t,x)$. The corresponding linearised Lagrangian density, expanded to second order in these fluctuations, is

$$\begin{aligned} \mathcal{L} = &\ \tfrac{1}{2\rho_s^2}\left(\partial_\mu \delta B\right)\left(\partial^\mu \delta B\right) - \tfrac{1}{2}e^2\left(\delta B\right)^2 + \tfrac{1}{2}\left(\partial_\mu \rho_s\right)\left(\partial^\mu \rho_s\right) + \left(\partial_\mu \rho_s\right)\left(\partial^\mu \delta\rho\right) \\ + &\ \tfrac{1}{2}\left(\partial_\mu \delta\rho\right)\left(\partial^\mu \delta\rho\right) - V(\rho_s) - \delta\rho V'(\rho_s) - \tfrac{1}{2}\left(\delta\rho\right)^2 V''(\rho_s)\,. \end{aligned}$$

Applying the variational principle in the fields δB and $\delta\rho$, the linearised Euler–Lagrange equations for these fields read

$$\partial_\mu\partial^\mu\delta\rho + V''(\rho_s)\delta\rho = 0, \qquad \partial_\mu\left[\frac{1}{\rho_s^2}\partial^\mu\delta B\right] + e^2\delta B = 0. \tag{7}$$

These equations being linear in the field variables, a normal mode expansion is warranted, with a spectrum of eigenfrequencies to be identified through the following *ansatz* for time dependence (the general solution then being constructed through linear combinations),

$$\delta\rho(t,x) = f(x)e^{-i\omega_1 t} + f^*(x)e^{i\omega_1 t}, \qquad \delta B(x,t) = g(x)e^{-i\omega_2 t} + g^*(x)e^{i\omega_2 t}. \tag{8}$$

If there exist solutions for which either of the eigenfrequencies ω_1 or ω_2 is pure imaginary, namely such that $\omega_i^2 < 0$ $(i = 1, 2)$, this would imply an exponential run-away time dependence for at least one of the linearised fluctuation modes, hence instability of the corresponding classical solution. In the above parametrisation of the normal modes, $f(x)$ and $g(x)$ stand for complex-valued functions on the circle, with $g(x)$ periodic and $f(x)$ periodic or anti-periodic according to the periodicity properties of the classical solution of which the stability needs to be established.

By direct substitution of the above *ansatz* into (8), the following eigenvalue equations are derived,

$$\left(-\frac{d^2}{dx^2} + V''(\rho_s)\right) f(x) = \omega_1^2\, f(x)\ , \tag{9}$$

$$\left[-\rho_s^2(x)\frac{d}{dx}\frac{1}{\rho_s^2(x)}\frac{d}{dx} + e^2\rho_s^2(x)\right] g(x) = \omega_2^2\, g(x)\ . \tag{10}$$

In the case of the vacuum configuration, $B = 0$, $\rho = \pm\rho_0$, the solutions to (9) and (10) are readily constructed through a Fourier series analysis on the circle for the unknown functions g and f, with $g(x) = 1/(2L)\sum_{n=-\infty}^{+\infty} e^{i\frac{\pi n}{L}x} g_n$ and $f(x) = 1/(2L)\sum_{n=-\infty}^{+\infty} e^{i\frac{\pi n}{L}x} g_n$. The spectrum is then found to be given as

$$\begin{aligned} f_n\ :&\qquad \omega_1^2 &=&\ \left(\frac{\pi n}{L}\right)^2 + M^2\rho_0^2,\\ g_n\ :&\qquad \omega_2^2 &=&\ \left(\frac{\pi n}{L}\right)^2 + e^2\rho_0^2, \end{aligned}$$

which is indeed positive definite. As it should, the vacuum configuration is indeed stable against all possible fluctuations in the fields.

For the non-trivial soliton configurations, the eigenvalue problem reads,

$$\left[\frac{d^2}{dy^2} - 6k^2\ \mathrm{sn}^2 y\right] f(y) = -\left[\frac{2\omega_1^2}{M^2\rho_0^2} + 1\right](1+k^2)f(y), \tag{11}$$

$$\left[-\frac{d^2}{dy^2} + W_B(y)\right]\psi(y) = \frac{2(1+k^2)\omega_2^2}{M^2\rho_0^2}\psi(y), \tag{12}$$

where

$$W_B(y) = 2\left[\frac{2e^2k^2}{M^2}\ \mathrm{sn}^2 y - \frac{1+k^2}{2} + \frac{1}{\mathrm{sn}^2 y}\right]. \tag{13}$$

In order to obtain (12), the following change of variable was introduced in (10), $g(x) = \rho_s(x)\psi(x)$. As a matter of fact, (11) is a Lamé equation, of which the solutions have been discussed and classified in Ref. [3], with the results listed in the Table below.

As may be seen from those results, the eigenfunctions $f(y) = \mathrm{sn}\,y\cdot\mathrm{cn}\,y$ and $f(y) = \mathrm{sn}\,y\cdot\mathrm{dn}\,y$ both have a positive eigenspectrum of ω_1^2 values and are both antiperiodic. However the last three solutions in the Table correspond to periodic functions, one of which is the zero mode associated to infinitesimal spatial translations, while among the other two there always exists one with a strictly negative eigenvalue ω_1^2. Consequently the solitons possessing a periodic boundary condition on the circle are unstable against fluctuations corresponding to that specific mode.

$f(y)$	ω_1^2
$\mathrm{sn}\, y \cdot \mathrm{cn}\, y$	$\frac{1}{2} M^2 \rho_0^2 \frac{3}{k^2+1}$
$\mathrm{sn}\, y \cdot \mathrm{dn}\, y$	$\frac{1}{2} M^2 \rho_0^2 \frac{3k^2}{1+k^2}$
$\mathrm{cn}\, y \cdot \mathrm{dn}\, y$	0
$\mathrm{sn}^2 y - \frac{1}{3k^2}(1 + k^2 \pm \sqrt{1 - k^2(1 - k^2)}$	$\frac{1}{2} M^2 \rho_0^2 \left(1 \mp 2 \frac{\sqrt{1-k^2(1-k^2)}}{1+k^2} \right)$

For what concerns the function $g(x)$ and its fluctuation spectral equation (12), the latter eigenvalue problem being equivalent to solving the Schrödinger equation for the potential $W_B(y)$, the issue may also be addressed from that point of view. Even though the Schrödinger equation may not be explicitly solve for that choice of potential, using the fact that the potential (13) is positive, implies in any case that the spectrum of ω_2^2 eigenvalues is likewise positive. No instability may arise in the $g(x)$, namely the B sector of fluctuations.

5 Concluding Remarks

The vacuum configuration has thus been confirmed to be stable against all fluctuations. Non-trivial static soliton configurations are also stable, but only in the sector of anti-periodic solitons and thus an anti-periodic boundary condition on the function $f(x)$. Periodic solitons, though, are always unstable.

Even though this work has some overlap with some previous studies, its originality lies with the fact that it considers specifically only the physical sector of the 1+1 dimensional abelian U(1) Higgs model without applying any gauge fixing procedure whatsoever. In particular the factorisation between the actual physical sector and the decoupled gauge variant sector offers two advantages. First, that no artefacts due to gauge fixing are introduced. Second, that potential spurious instabilities lying solely within the gauge variant and unphysical sector are avoided from the outset, a feature any other approach having been developed so far cannot achieve. All these approaches have until now relied on some gauge fixing procedure rather than applying such a gauge invariant and physical factorisation. Hence these approaches runs the risk of identifying instabilities which in fact are pure gauge and thus unphysical. Such difficulties are avoided from the outset within our approach.

When the potential $V(\rho)$ is taken to be the Higgs potential, as we did, the static solutions in ρ with $B = 0$ are in fact those of Refs. [4, 5]. These authors also solve the model of a single complex scalar field on the circle, while in Ref. [6], the authors consider the $1 + 1$ abelian Higgs Model on the circle, but in contradistinction to what we have done, they apply a gauge fixing procedure in terms of the gauge field components A_0 and A_1. We recover the same solutions on the circle. Our work has established that the soliton configurations with anti-periodic boundary conditions are stable. In Refs. [5, 6], the static soliton solutions with periodic boundary conditions are also found to be unstable.

A possible continuation of the present study would be the computation of the complete spectrum of fluctuation eigenfrequencies for the stable and unstable classical soliton configurations, inclusive of the electromagnetic sector contributions but retaining only the physical degrees of freedom, in order to determine the quantum corrections of the mass spectrum of these solitonic field configurations of the 1+1 dimensional abelian U(1) Higgs model, whether on the circle or on the real line.

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